Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 4, 1999

STABILITY OF A MULTILAYERED COMPOSITE CONICAL SHELL UNDER UNIFORM EXTERNAL PRESSURE

A. N. Andreyev

UDC 539.3

The stability of equilibrium of a layered composite circular conical truncated shell loaded with uniform external pressure is investigated. A parametric analysis of the critical pressure intensities is carried out with allowance for the transverse shear, the moment character of the subcritical state of equilibrium, and the subcritical strains.

1. Linearized Differential Equations of Stability of a Multilayered Orthotropic Conical Shell. We consider an orthotropic circular conical truncated shell of thickness h which consists of m composite fibrous layers. Let 2α be the cone angle, $s = x^1$ be the distance measured along the generatrix of the cone from its top ($0 < a \leqslant s \leqslant b$), and $\varphi = x^2$ be the angular coordinate ($-\pi \leqslant \varphi \leqslant \pi$). The Lamé parameters A_1 and A_2 and the curvature radii R_1 and R_2 of the coordinate lines have the form

$$A_1 = 1, \quad A_2 = s\sin\alpha, \quad R_1 = \infty, \quad R_1 = s\tan\alpha. \tag{1.1}$$

We confine ourselves to the case where the direction of the orthotropy axes coincides with the direction of the coordinate axes, and the structural reinforcement parameters of all the layers of the shell are independent of the angular coordinate φ but can depend on the meridional coordinate s, which is the case, for example, where the layers of the shell are reinforced in the circumferential and meridional directions by fibers of constant cross section. Assuming that the shell is sufficiently thin, we ignore quantities of order h/R_2 compared to unity in all equations.

The stability analysis of the shell is based on the nonclassical equations [1] which take transverse shear strains into account. Passing from the tensorial to the physical components and from the covariant to the partial derivatives in the tensor equations [1] and bearing in mind (1.1), we arrive at a closed system of linearized differential stability equations of a conical shell. The system comprises the following groups of relations:

— Relations of elasticity

$$\sigma_{11}^{(k)} = a_{11}^{(k)}\varepsilon_{11}^{(k)} + a_{12}^{(k)}\varepsilon_{22}^{(k)}, \quad \sigma_{22}^{(k)} = a_{12}^{(k)}\varepsilon_{11}^{(k)} + a_{22}^{(k)}\varepsilon_{22}^{(k)}, \quad \sigma_{12}^{(k)} = a_{33}^{(k)}\gamma_{12}^{(k)},$$
$$\tau_{13}^{(k)} = c_{11}^{(k)}\gamma_{13}^{(k)}, \quad \tau_{23}^{(k)} = c_{22}^{(k)}\gamma_{23}^{(k)}; \tag{1.2}$$

— The distribution law for variation in the physical components of the displacement vector over the thickness of the package of layers

$$v_1^{(k)} = u_1 - z\,\frac{\partial w}{\partial s} + \mu_{11}^{(k)}\pi_1, \qquad v_2^{(k)} = u_2 - \frac{z}{A_2}\,\frac{\partial w}{\partial \varphi} + \mu_{22}^{(k)}\pi_2, \qquad v_3^{(k)} = w, \tag{1.3}$$

$$\mu_{\lambda\lambda}^{(k)} = \frac{f(z) - f(h_{k-1})}{c_{\lambda\lambda}^{(k)}} + \sum_{j=1}^{k-1}\frac{f(h_j) - f(h_{j-1})}{c_{\lambda\lambda}^{(j)}}, \qquad \lambda = 1,\, 2;$$

Kemerovo State University, Kemerovo 650043. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 4, pp. 198–207, July–August, 1999. Original article submitted September 11, 1997.

— Relations between the variations in the strains and the physical components of the displacement vector

$$\varepsilon_{11}^{(k)} = \frac{\partial u_1}{\partial s} - z\,\frac{\partial^2 w}{\partial s^2} + \mu_{11}^{(k)}\,\frac{\partial \pi_1}{\partial s} + \frac{\partial \mu_{11}^{(k)}}{\partial s}\,\pi_1 + \frac{\partial \tilde{w}}{\partial s}\,\frac{\partial w}{\partial s},$$

$$\varepsilon_{22}^{(k)} = \frac{1}{A_2}\Big[\frac{\partial u_2}{\partial \varphi} - \frac{z}{A_2}\,\frac{\partial^2 w}{\partial \varphi^2} + \mu_{22}^{(k)}\,\frac{\partial \pi_2}{\partial \varphi} + \sin\alpha\Big(u_1 - z\,\frac{\partial w}{\partial s} + \mu_{11}^{(k)}\pi_1\Big)\Big] + \frac{w}{R_2} + \frac{1}{A_2}\,\frac{\partial \tilde{w}}{\partial \varphi}\,\frac{1}{A_2}\,\frac{\partial w}{\partial \varphi},$$

$$\gamma_{12}^{(k)} = \frac{1}{A_2}\Big[\frac{\partial u_1}{\partial \varphi} - z\frac{\partial^2 w}{\partial s\,\partial \varphi} + \mu_{11}^{(k)}\,\frac{\partial \pi_1}{\partial \varphi} - \sin\alpha\Big(u_2 - \frac{z}{A_2}\,\frac{\partial w}{\partial \varphi} + \mu_{22}^{(k)}\pi_2\Big)\Big] \tag{1.4}$$

$$+\frac{\partial u_2}{\partial s} - z\,\frac{\partial}{\partial s}\Big(\frac{1}{A_2}\,\frac{\partial w}{\partial \varphi}\Big) + \mu_{22}^{(k)}\,\frac{\partial \pi_2}{\partial s} + \frac{\partial \mu_{22}^{(k)}}{\partial s}\,\pi_2 + \frac{1}{A_2}\,\frac{\partial \tilde{w}}{\partial s}\,\frac{\partial w}{\partial \varphi} + \frac{1}{A_2}\,\frac{\partial \tilde{w}}{\partial \varphi}\,\frac{\partial w}{\partial s},$$

$$\gamma_{13}^{(k)} = \frac{f'(z)}{c_{11}^{(k)}}\,\pi_1, \qquad \gamma_{23}^{(k)} = \frac{f'(z)}{c_{22}^{(k)}}\,\pi_2;$$

— Relations between the variations in the generalized internal forces and the moments in the shell surface and the variations in the internal stresses in its layers

$$\|T_{\beta\lambda}, M_{\beta\lambda}, S_{\beta\lambda}\| = \sum_{k=1}^{m}\int_{h_{k-1}}^{h_k} \sigma_{\beta\lambda}^{(k)}\|1, z, \mu_{\lambda\lambda}^{(k)}\|\,dz,$$

$$Q_\lambda = \sum_{k=1}^{m}\int_{h_{k-1}}^{h_k}\Big[\sigma_{1\lambda}^{(k)}\,\frac{\partial \mu_{\lambda\lambda}^{(k)}}{\partial s} + \sigma_{2\rho}^{(k)}\,\frac{\sin\alpha}{A_2}\,(\mu_{11}^{(k)} - \mu_{22}^{(k)}) + \tau_{\lambda 3}^{(k)}\,\frac{f'(z)}{c_{\lambda\lambda}^{(k)}}\Big]\,dz, \quad \beta,\ \lambda,\ \rho = 1,\ 2, \quad \lambda \neq \rho; \tag{1.5}$$

— Differential equations of neutral equilibrium of a conical shell for variations in the generalized forces and moments

$$\frac{\partial}{\partial s}\,(A_2T_{11}) - T_{22}\sin\alpha + \frac{\partial T_{21}}{\partial \varphi} = 0, \qquad \frac{\partial T_{22}}{\partial \varphi} + T_{21}\sin\alpha + \frac{\partial}{\partial s}\,(A_2T_{12}) = 0,$$

$$\frac{\partial^2}{\partial s^2}\,(A_2M_{11}) - \sin\alpha\,\frac{\partial M_{22}}{\partial s} + 2\,\frac{\partial^2 M_{12}}{\partial s\,\partial \varphi} + \frac{1}{A_2}\,\frac{\partial^2 M_{22}}{\partial \varphi^2} + 2\,\frac{\sin\alpha}{A_2}\,\frac{\partial M_{21}}{\partial \varphi} - \frac{A_2T_{22}}{R_2}$$

$$+\frac{\partial}{\partial s}\Big(A_2\tilde{T}_{11}\,\frac{\partial w}{\partial s} + A_2T_{11}\,\frac{\partial \tilde{w}}{\partial s} + \tilde{T}_{21}\,\frac{\partial w}{\partial \varphi} + T_{21}\,\frac{\partial \tilde{w}}{\partial \varphi}\Big) \tag{1.6}$$

$$+\frac{\partial}{\partial \varphi}\Big(\tilde{T}_{12}\,\frac{\partial w}{\partial s} + T_{12}\,\frac{\partial \tilde{w}}{\partial s} + \frac{\tilde{T}_{22}}{A_2}\,\frac{\partial w}{\partial \varphi} + \frac{T_{22}}{A_2}\,\frac{\partial \tilde{w}}{\partial \varphi}\Big) = 0,$$

$$\frac{\partial}{\partial s}\,(A_2S_{11}) - S_{22}\sin\alpha + \frac{\partial S_{21}}{\partial \varphi} - A_2Q_1 = 0, \qquad \frac{\partial S_{22}}{\partial \varphi} + S_{21}\sin\alpha + \frac{\partial}{\partial s}\,(A_2S_{12}) - A_2Q_2 = 0.$$

Here the brackets at the subscripts of the physical components of the vectors and the tensors are dropped, the tilde sign is used to denote the characteristics of the subcritical state, and z is the normal coordinate reckoned from the inner shell surface:

$$z \in [0, h] = \bigcup_{k=1}^{m}\,[h_{k-1}, h_k],$$

where kth layer of the shell ($k = 1, 2, \ldots, m$) corresponds to the interval $[h_{k-1}, h_k]$ of variation of this coordinate. We write the function $f(z)$ in the form

$$f(z) = z^3 - 1.5hz^2, \tag{1.7}$$

which corresponds to the quadratic dependence of the transverse shear stress on z [1, 2].

Equations (1.1)–(1.7) constitute the complete system of nonclassical differential equations of the stability problem of the conical shell. The order of this system is 12, which implies that six boundary conditions should be specified at the boundary of the domain. If the shell is closed in the circumferential direction and its ends are rigidly fixed, these conditions require the 2π-periodicity of the solution in the φ coordinate and vanishing of the generalized displacements at the clamped sections [1, 2]:

$$w = \frac{\partial w}{\partial s} = u_1 = u_2 = \pi_1 = \pi_2 = 0 \quad \text{for} \quad s = a, \quad s = b. \tag{1.8}$$

Equations (1.1)–(1.7) take into account the orthotropy of the deformability properties, the low shear rigidity of all or part of the layers, the moment nature of the subcritical state, and the subcritical strains. Therefore, they are applicable to analysis of the stability of equilibrium of a thin-walled layered composite conical shell for the general loading and boundary conditions. One of the advantages of these equations is that their order and structure are independent of the number of shell layers and the structure of the layer package, which simplifies the formulation and investigation of the stability problem of a multilayered shell as an eigenvalue problem for a linear system of partial differential equations. The coefficients $\tilde{T}_{\beta\lambda}$, $\partial\tilde{w}/\partial s$, and $\partial\tilde{w}/\partial\varphi$ in this system depend on the parameter of external loads and can be determined by integration of the corresponding linear or nonlinear static boundary-value problem. The nonclassical static equations of a conical shell can be obtained from the general equations [2] similarly to the derivation of the stability equations of the shell. The linearized variant of these equation follows from Eqs. (1.1)–(1.7) if the parametric terms denoted by tilde sign are ignored in (1.4) and the fictitious load is replaced by the actual load in (1.6). These linearized differential equations are given in Sec. 2.

To estimate the effect of the transverse shear strains on the critical stability parameters, we use the limit passage [1, 2]

$$c_{\beta\beta}^{(k)} \to \infty \tag{1.9}$$

($k = 1, 2, \ldots, m$ and $\beta = 1, 2$) from Eqs. (1.1)–(1.7) to the classical stability equations of a conical shell.

We introduce the variables

$$\begin{gathered} x = \frac{s}{b}, \qquad \lambda = \frac{P}{E_1^c}, \qquad w = hy_1, \qquad y_2 = \frac{\partial y_1}{\partial x}, \qquad u_1 = by_3, \\ u_2 = by_4, \qquad \pi_1 = E_1^c bh^{-3} y_5, \qquad \pi_2 = E_1^c bh^{-3} y_6, \\ \frac{\partial}{\partial s}(A_2 M_{11}) - M_{22}\sin\alpha + 2\frac{\partial M_{12}}{\partial\varphi} + A_2\tilde{T}_{11}\frac{\partial w}{\partial s} + A_2 T_{11}\frac{\partial\tilde{w}}{\partial s} + \tilde{T}_{21}\frac{\partial w}{\partial\varphi} + T_{21}\frac{\partial\tilde{w}}{\partial\varphi} = h^2 E_1^c y_7, \\ A_2 M_{11} = h^2 bE_1^c y_8, \ \ A_2 T_{11} = hbE_1^c y_9, \ \ A_2 T_{12} = hbE_1^c y_{10}, \ \ A_2 S_{11} = h^4 by_{11}, \ \ A_2 S_{12} = h^4 by_{12}, \end{gathered} \tag{1.10}$$

where x is the dimensionless independent variable ($a/b \leqslant x \leqslant 1$), P and λ are the dimensional and dimensionless load parameter, respectively, $\boldsymbol{y} = [y_1, y_2, \ldots, y_{12}]^t$ is the column vector containing the dimensionless kinematic and force characteristics of the stress-strain state of the shell, and E_1^c is the Young's modulus of the binding material of the first layer.

The differential stability equations (1.1)–(1.7) and the boundary conditions (1.8) can be written in variables (1.10) in matrix form

$$A(x, D_\varphi)\frac{\partial \boldsymbol{y}}{\partial x} = B(x, D_\varphi)\boldsymbol{y} + \lambda C(x, D_\varphi)\boldsymbol{y}; \tag{1.11}$$

$$\|E_6, O_6\|\boldsymbol{y}(a/b, \varphi) = \boldsymbol{0}, \qquad \|E_6, O_6\|\boldsymbol{y}(1, \varphi) = \boldsymbol{0}. \tag{1.12}$$

Here E_6 and O_6 are the 6×6 identity and null matrices; A, B, and C are the 12×12 matrices whose elements are the polynomials of the differential operator D_φ ($D_\varphi = \partial/\partial\varphi$) with coefficients depending on the variable x. For the nonaxisymmetric subcritical state of equilibrium, the matrix elements of the parametric terms C depend also on the angular coordinate φ. The expressions for the elements of matrices A, B, and C are not given here in view of their cumbersome form. We only indicate the zero and nonzero columns of the matrix C, integrating the numbers of the zero columns into the set K and the numbers of the nonzero columns into the set J [3]. Depending on whether the subcritical strains are taken into account or not, the number of the zero columns of the matrix C is equal to 8 or 10:

$$J = \{1,2,9,10\}, \qquad K = \{1,2,\dots,12\} - J; \tag{1.13}$$

$$J = \{1,2\}, \qquad K = \{1,2,\dots,12\} - J. \tag{1.14}$$

It is worth noting that deletion of the 5th, 6th, 11th, and 12th rows and columns in the (12×12) matrices A, B, and C results in the corresponding (8×8) matrices of the coefficients of the classical system of differential equations of the stability problem of a conical shell. This follows from the limit transition (1.9), since the elements of the above-mentioned rows and columns of the matrices A, B, and C vanish as $c_{\beta\beta}^{(k)} \to \infty$.

2. Stability of an Orthotropic Layered Conical Shell Under Uniform External Pressure. We investigate the stability of a multilayered orthotropic circular conical truncated rigidly fixed shell loaded with a uniform external pressure P. In this case, the stress-strain state of equilibrium is the axisymmetric state, and the angular component $\tilde{v}_2^{(k)}$ of the displacement vector and the quantities related to it vanish, which simplifies the parametric terms of the equations by setting

$$\tilde{T}_{12} = \tilde{T}_{21} = \frac{\partial \tilde{w}}{\partial \varphi} = 0. \tag{2.1}$$

The determination of the stress-strain state of equilibrium of the shell is also simplified and reduces to integration of a system of ordinary differential equations for appropriate boundary conditions. In a linear approximation, this system comprises the following groups of relations:

— Relations of elasticity

$$\tilde{\sigma}_{11}^{(k)} = a_{11}^{(k)}\tilde{\varepsilon}_{11}^{(k)} + a_{12}^{(k)}\tilde{\varepsilon}_{22}^{(k)}, \quad \tilde{\sigma}_{22}^{(k)} = a_{12}^{(k)}\tilde{\varepsilon}_{11}^{(k)} + a_{22}^{(k)}\tilde{\varepsilon}_{22}^{(k)}, \quad \tilde{\tau}_{13}^{(k)} = c_{11}^{(k)}\tilde{\gamma}_{13}^{(k)}; \tag{2.2}$$

— The distribution law of the physical components of the displacement vector over the thickness of the package of layers

$$\tilde{v}_1^{(k)} = \tilde{u}_1 - z\,\frac{d\tilde{w}}{ds} + \mu_{11}^{(k)}\tilde{\pi}_1, \qquad \tilde{v}_3^{(k)} = \tilde{w}; \tag{2.3}$$

— Strain–displacement relations

$$\begin{gathered} \tilde{\varepsilon}_{11}^{(k)} = \frac{d\tilde{u}_1}{ds} - z\,\frac{d^2\tilde{w}}{ds^2} + \mu_{11}^{(k)}\,\frac{d\tilde{\pi}_1}{ds} + \frac{\partial \mu_{11}^{(k)}}{\partial s}\,\tilde{\pi}_1, \\ \tilde{\varepsilon}_{22}^{(k)} = \frac{\sin\alpha}{A_2}\Big[\tilde{u}_1 - z\,\frac{d\tilde{w}}{ds} + \mu_{11}^{(k)}\tilde{\pi}_1\Big] + \frac{\tilde{w}}{R_2}, \qquad \tilde{\gamma}_{13}^{(k)} = \frac{f'(z)}{c_{11}^{(k)}}\,\tilde{\pi}_1; \end{gathered} \tag{2.4}$$

— Relations (1.5) between the generalized internal forces and moments in the shell surface and the internal stresses in its layers;

— Differential equations of equilibrium of the shell for the forces and moments

$$\begin{gathered} \frac{d}{ds}\,(A_2\tilde{T}_{11}) - \tilde{T}_{22}\sin\alpha = 0, \qquad \frac{d}{ds}\,(A_2\tilde{S}_{11}) - \tilde{S}_{22}\sin\alpha - A_2\tilde{Q}_1 = 0, \\ \frac{d^2}{ds^2}\,(A_2\tilde{M}_{11}) - \frac{d\tilde{M}_{22}}{ds}\sin\alpha - \frac{A_2\tilde{T}_{22}}{R_2} = A_2 P. \end{gathered} \tag{2.5}$$

The function $f(z)$ has the form (1.7). We introduce the variables

$$\begin{gathered} s = bx, \quad \tilde{w} = \frac{Ph\tilde{y}_1}{E_1^c}, \quad \tilde{y}_2 = \frac{d\tilde{y}_1}{dx}, \quad \tilde{u}_1 = \frac{Pb\tilde{y}_3}{E_1^c}, \quad \tilde{\pi}_1 = Pbh^{-3}\tilde{y}_4, \\ \frac{d}{ds}\,(A_2\tilde{M}_{11}) - \tilde{M}_{22}\sin\alpha = Ph^2\tilde{y}_5, \quad A_2\tilde{M}_{11} = Ph^2 b\tilde{y}_6, \quad A_2\tilde{T}_{11} = Phb\tilde{y}_7, \quad A_2\tilde{S}_{11} = \frac{Ph^4 b\tilde{y}_8}{E_1^c}, \end{gathered} \tag{2.6}$$

where $\tilde{\boldsymbol{y}} = [\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_8]^t$ is the column vector of the dimensionless kinematic and force characteristics of the stress-strain state of the shell, x is the dimensionless independent variable $(a/b \leqslant x \leqslant 1)$, and E_1^c is the

Young's modulus of the binding material of the first layer.

Using the variables (2.6), we write the system of differential equations of axisymmetric bending of a conical shell and the corresponding boundary conditions in matrix form

$$\tilde{\boldsymbol{y}}'(x) = A(x)\tilde{\boldsymbol{y}}(x) + \boldsymbol{f}(x), \quad \|E_4, O_4\|\tilde{\boldsymbol{y}}(a/b) = \boldsymbol{0}, \quad \|E_4, O_4\|\tilde{\boldsymbol{y}}(1) = \boldsymbol{0}. \tag{2.7}$$

Here $A(x)$ is an 8×8 matrix, E_4 and O_4 are the 4×4 identity and null matrices, and $\boldsymbol{f}(x)$ is the 8-dimensional vector. The expressions for the matrix $A(x)$ and the vector $\boldsymbol{f}(x)$ are not given here because of their cumbersome form [if necessary, these can be obtained from (1.1), (1.5), (1.7), and (2.2)–(2.7)].

A numerical analysis shows that the spectral structure of the matrix $A(x)$ is, on the whole, similar to that of the matrix containing the coefficients of the nonclassical system of differential equations of axisymmetric bending of a cylindrical shell, which is described in [4]. The strong instability of these differential equations, which requires special numerical algorithms to integrate the boundary-value problem of bending, shows up in problem (2.7) as well. In the example considered below, the numerical solution of the problem was obtained by the invariant imbedding method [5, 6].

One avoids integration of the boundary-value problem (2.7) if the stability of a shell is analyzed in a simplified formulation where the subcritical strains and the moment nature of the subcritical state of equilibrium are ignored. In this approximation, the subcritical angles of rotation of the normal are set equal to zero:

$$\frac{\partial \tilde{w}}{\partial s} = 0, \tag{2.8}$$

while the subcritical forces are determined from the formulas

$$\tilde{T}_{11} = -PR_2(1 - a^2/s^2)/2, \qquad \tilde{T}_{22} = -PR_2, \tag{2.9}$$

which are obtained by integration of the momentless equations of equilibrium of a conical shell (see, e.g., [7]). We emphasize that, in this case, the equalities (2.1) hold rigorously, and the equalities (2.8) and (2.9) are the simplifying assumptions. The error introduced by these assumptions into the determination of the critical stability parameters is studied below.

Thus, the characteristics of the subcritical state of equilibrium have been determined and the matrix of the parametric terms C has been formed, its elements being independent of the angular variable φ. We search for a solution of the boundary-value problem (1.11) and (1.12) in the form of a trigonometric Fourier series

$$\boldsymbol{y} = \sum_{n=-\infty}^{\infty} \boldsymbol{y}_n(x) \exp(in\varphi) \tag{2.10}$$

($i = \sqrt{-1}$) with vector coefficients $\boldsymbol{y}_n(x)$. Obviously, the form of solution (2.10) satisfies the 2π-periodicity condition in the φ coordinate. Substituting the expansion (2.10) into Eqs. (1.11) and the boundary conditions (1.12) and separating the angular variable, we obtain a series of linear boundary-value eigenvalue problems for the systems of ordinary differential equations

$$\boldsymbol{y}_n'(x) = A_n(x)\boldsymbol{y}_n(x) + \lambda B_n(x)\boldsymbol{y}_n(x), \qquad \|E_6, O_6\|\boldsymbol{y}_n(a/b) = \|E_6, O_6\|\boldsymbol{y}_n(1) = \boldsymbol{0}. \tag{2.11}$$

The elements of the matrices $A_n(x)$ and $B_n(x)$ can be obtained from the elements of the matrices A, B, and C with the use of the transformation

$$D_\varphi \to in. \tag{2.12}$$

Since the solution $\boldsymbol{y}(x, \varphi)$ of problem (1.11), (1.12) is real, the coefficients $\boldsymbol{y}_n(x)$ and $\boldsymbol{y}_{-n}(x)$ of the expansion (2.10) are complex conjugate: $\boldsymbol{y}_{-n}(x) = \bar{\boldsymbol{y}}_n(x)$. Therefore, it suffices to consider the boundary-value problems (2.11) only for $n \geqslant 0$. It should be noted that, in the equations of stability (1.11), the odd degrees of the operator D_φ act on the 4th, 6th, 10th, and 12th components of the vector $\boldsymbol{y}$, and the even degrees of the operator act on other components. With allowance for (2.12), it follows that problems (2.11) reduce to real problems by using the transformation $y_n^p \to iy_n^p$ ($p = 4$, 6, 10, and 12 enumerates the components of the vector $\boldsymbol{y}_n$).

The dimensionless critical pressure intensity λ_{n_0} and the vector function $\boldsymbol{y}^*$, which determines the buckling shape of the shell, are calculated from the formulas

$$\lambda_{n_0} = \inf_{n \geqslant 0} \lambda_n, \qquad \boldsymbol{y}^* = \boldsymbol{y}^*_{n_0}(x) \exp(in_0\varphi) + \bar{\boldsymbol{y}}^*_{n_0}(x) \exp(-in_0\varphi),$$

where λ_n and $\boldsymbol{y}^*_n(x)$ are the minimal eigenvalue and the corresponding vector eigenfunction for the nth boundary-value problem (2.11).

The numerical solution of problems (2.11) was obtained by the method of [3, 5] with the use of the orthonormal coordinate system

$$\boldsymbol{Y}_{kj}(x) = \sqrt{\frac{2k-1}{1-a/b}}\, P_{k-1}\Big(2\frac{x-a/b}{1-a/b} - 1\Big)\boldsymbol{e}_j \qquad (k = 1, 2, \ldots, L; \quad j \in J), \tag{2.13}$$

where $P_k(t)$ are the Legendre polynomials orthogonal in the segment $[-1; 1]$ and $\boldsymbol{e}_j$ are the vectors of the standard orthonormal basis in $\mathbb{R}^{12}$. It follows from (1.13) and (1.14) that the coordinate system (2.13) consists of $4L$ or $2L$ vectors, depending on whether the subcritical strains are taken into account or not. Therefore, to determine the eigenvalues and eigenvectors of problems (2.11) by the method of [3, 5], one needs to integrate the boundary-value problems for $12 \times 4L$ matrices and solve the algebraic eigenvalue problem for $4L \times 4L$ matrices in the first case and to integrate the boundary-value problems for $12 \times 2L$ matrices and solve the algebraic eigenvalue problem for $2L \times 2L$ matrices in the second case. The boundary-value problems for matrix differential equations were solved by the invariant imbedding method [5], and the QR-algorithm was used in combination with reduction of the matrix to the Hessenberg form [8] to determine the eigenvalues. The value of L which ensures high accuracy of the result was found by a numerical investigation of the convergence rate of the method. The calculations were carried out on an "Elbrus-2" computer.

3. Numerical Results. We introduce the following notation: P_1^* is the critical pressure found on the basis of the classical stability equations of a conical shell without allowing for the subcritical strains and the moment nature of the subcritical state of equilibrium, P_2^* is the critical pressure found on the basis of the nonclassical stability equations ignoring these factors, P_3^* is the critical pressure found on the basis of Eqs. (1.11) with inclusion of the moment nature of the subcritical state but ignoring the subcritical strains, and P_4^* is the critical pressure found on the basis of Eqs. (1.11) with inclusion of the moment nature of the subcritical state and the subcritical strains.

Table 1 lists the data which characterize the convergence rate of the method relative to the parameter L. The first column contains the values of this parameter, and the second and third columns contain, respectively, the corresponding values of the critical pressure P_3^* and the number of waves in the circumferential direction n_0. The results were obtained for a two-layered shell whose first (inner) layer was reinforced by fibers of constant cross section in the circumferential direction, and whose second layer was reinforced in the meridional direction. The shell characteristics are as follows:

— the geometrical parameters are

$$\alpha = \pi/8, \quad a/b = 0.2, \quad h/b = 0.33, \quad h_1 - h_0 = h_2 - h_1 = 0.5h; \tag{3.1}$$

— the mechanical parameters are

$$E_1^c/E_2^c = 1, \quad E_1^c/E_1^a = E_2^c/E_2^a = 0.05, \quad \nu_1^c = \nu_2^c = \nu_1^a = \nu_2^a = 0.3; \tag{3.2}$$

— the structural parameters are

$$\omega_{z1} = \omega_{z2} = 0.5, \qquad \omega_1 = 0.5, \qquad \omega_2\Big|_{x=a/b} = 0.9. \tag{3.3}$$

Here $h_k - h_{k-1}$, ω_k and ω_{zk} are the thickness of the kth layer and the reinforcement intensity in its surface and over the thickness [9], and E_k^c, ν_k^c, E_k^a, and ν_k^a ($k = 1, 2$) are the Young's modulus and the Poisson ratio of the binding material and the reinforcing fibers of the kth layer, respectively. The effective rigidities and

TABLE 1

L	$10^2 P_3^*/E_1^c$	n_0
3	0.765	3
4	0.723	3
5	0.711	3
6	0.704	3
7	0.704	3
8	0.704	3
9	0.704	3

TABLE 2

b/h	$10^2 P_1^*/E_1^c$	$10^2 P_2^*/E_1^c$	$10^2 P_3^*/E_1^c$	$10^2 P_4^*/E_1^c$	n_0
150	0.390	0.343	0.411	0.432	8
200	0.175	0.163	0.178	0.181	8
250	0.093	0.088	0.092	0.093	9
300	0.054	0.052	0.053	0.053	9
350	0.036	0.035	0.035	0.035	9

TABLE 3

E^a/E^c	$10^2 P_1^*/E_1^c$	$10^2 P_2^*/E_1^c$	$10^2 P_3^*/E_1^c$	$10^2 P_4^*/E_1^c$	n_0
10	0.137	0.129	0.142	0.145	8
20	0.175	0.163	0.178	0.181	8
30	0.203	0.187	0.203	0.206	8
40	0.225	0.206	0.221	0.224	8
50	0.246	0.224	0.238	0.241	8
60	0.265	0.240	0.254	0.257	8

compliances of the layers were determined from the equations for the structural model of a reinforced layer [9].

One can see from Table 1 that the process of calculation is stabilized even for $L = 6$. This value or close values of L correspond not only to the parameters (3.1)–(3.3), but also to other values of the shell parameters considered below. A further numerical analysis is performed for $L = 8$.

Table 2 lists the values of the critical pressure for a two-layered composite shell, whose inner layer is reinforced by fibers of constant cross section in the circumferential direction and whose outer layer is reinforced in the meridional direction, depending on b/h. The results were obtained for $\alpha = \pi/20$ and $a/b = 0.9$; the other parameters had the values (3.1)–(3.3). One can see from Table 2 that neglect of the transverse shear leads to overestimation of the critical pressure values, and neglect of the moment nature of the subcritical state and the subcritical strains results in underestimation of the critical pressure values. The relative errors introduced into the determination of the critical pressure by ignoring these factors attain a maximum for $b/h = 150$ and they are 12.05, 20.12, and 4.85%, respectively. The effect of all the parameters under consideration becomes weaker as the parameter b/h increases and almost vanishes for $b/h = 350$.

Table 3 lists the values of the critical pressure for a two-layered composite shell, whose inner layer is reinforced by fibers of constant cross section in the circumferential direction and whose outer layer is reinforced in the meridional direction, depending on the parameter E^a/E^c ($E_1^a = E_2^a \equiv E^a$ and $E_1^c = E_2^c \equiv E^c$). The results were obtained for $b/h = 200$, and the values of other parameters are given in Table 1. It follows from Table 3 that, as the parameter E^a/E^c increases, which is accompanied by an increase of the transverse shear compliance of the shell layers [9], the effect of the transverse shear strains on the values of the critical pressure becomes more pronounced, while the effect of the subcritical strains and the moment nature of the state of equilibrium becomes weaker. For example, the relative error introduced in the determination of the critical pressure when the transverse shear strains (the moment nature of the subcritical state of equilibrium) are not taken into account is 5.84% (10.08%) for $E^a/E^c = 10$ and 9.43% (5.83%) for $E^a/E^c = 60$. It should be noted that the weakening of the influence of the moment nature of the subcritical state on the critical stability parameters for pliable shells with respect to the transverse shear was revealed in [10, 11], where the stability of a cylindrical shell under external pressure was studied.

Table 4 lists the values of the critical pressures $P_1^*, \ldots, P_4^*$ for a three-layered composite shell of a symmetrical structure consisting of isotropic layers versus the parameter E_1/E_2. The results were obtained

TABLE 4

E_1/E_2	$10^3\,P_1^*/E_1$	$10^3\,P_2^*/E_1$	$10^3\,P_3^*/E_1$	$10^3\,P_4^*/E_1$	n_0
10	1.148	0.863	0.993	1.008	5
15	1.106	0.755	0.866	0.879	5
20	1.087	0.684	0.780	0.790	5
25	1.075	0.627	0.709	0.718	5
30	1.066	0.579	0.649	0.656	5
35	1.061	0.542	0.604	0.611	5
40	1.056	0.505	0.562	0.568	6
45	1.053	0.471	0.520	0.525	6

TABLE 5

b/h	$10^3\,P_1^*/E_1$	$10^3\,P_2^*/E_1$	$10^3\,P_3^*/E_1$	$10^3\,P_4^*/E_1$	n_0
15	1.487	0.453	0.485	0.491	3
20	0.659	0.286	0.301	0.305	3
25	0.345	0.189	0.197	0.199	3
30	0.199	0.128	0.132	0.134	3
35	0.133	0.095	0.097	0.098	3

for the following parameters:

$$\frac{b}{h} = 50, \quad \frac{a}{b} = 0.7, \quad \alpha = \frac{\pi}{10}, \quad \frac{E_1}{E_3} = 1, \quad t_1 = t_3 = 0.1h, \quad t_2 = 0.8h, \quad \nu_1 = \nu_2 = \nu_3 = 0.3. \tag{3.4}$$

Here E_k, ν_k, and t_k are the Young's modulus, the Poisson ratio, and the thickness of the kth layer, respectively. Table 4 shows that the omission of the transverse shear leads to overestimation of the critical pressures and the omission of the moment nature of the subcritical state and the subcritical strains results in underestimation of the critical pressures. The relative error introduced into the determination of the critical pressure when the transverse shear is ignored increases from 24.82 to 55.27% as the parameter E_1/E_2 increases from 10 to 45. This great error shows that it is necessary to take into account the transverse shear in solving the stability problems of shells with significantly different rigidities of the layers. When the moment nature of the subcritical state and the subcritical strains are not taken into account, the relative errors decrease, respectively, from 15.06 and 1.51% to 10.40 and 0.96% as the parameter E_1/E_2 increases from 10 to 45.

Table 5 lists the values of the critical pressures for a three-layered composite shell of a symmetrical structure consisting of homogeneous isotropic layers versus b/h. The results were obtained for $a/b = 0.2$ and $E_1/E_2 = 20$ with the values of other parameters given by (3.4). As in the previous case, of the three factors investigated, namely, the transverse shear strains, the moment nature of the subcritical state, and the subcritical strains, the transverse shear strains are the most important factor. Indeed, the relative error when the shear is ignored is 69.54%, while the errors when the moment nature of the subcritical state and the subcritical strains are not taken into account are 7.06 and 1.24%, respectively, for $b/h = 15$. As the parameter b/h increases, the relative errors due to neglect of these factors decrease and they are 28.57, 2.10, and 1.03%, respectively, for $b/h = 35$.

We note that the resulting relationships between the critical pressure and the moment nature of the subcritical state and the transverse shear are similar to those obtained in [10] for a cylindrical shell.

REFERENCES

1. A. N. Andreyev and Yu. V. Nemirovskii, "Stability of elastic multilayered reinforced shells," *Mekh. Kompozit. Mat.*, No. 1, 86–95 (1979).
2. A. N. Andreyev and Yu. V. Nemirovskii, "Theory of elastic multilayered anisotropic shells," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 5, 87–96 (1977).
3. A. N. Andreyev, "Free vibrations of layered elastic composite shells of revolution," *Prikl. Mekh. Tekh. Fiz.*, **36**, No. 5, 145–154 (1995).
4. A. N. Andreyev, "Estimation of the strength of an elastic layered composite shell of revolution in a geometrically nonlinear formulation," *Prikl. Mekh.*, **26**, No. 7, 43–49 (1990).
5. A. N. Andreyev, "Numerical solution of linear boundary-value stability problems of layered shells of revolution," *Prikl. Mekh.*, **25**, No. 8, 60–66 (1989).
6. A. N. Andreyev, "Numerical integration of the equations of axisymmetrical bending of layered shells of revolution by the invariant imbedding method," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, **73** (1985), pp. 137–148.
7. É. I. Grigolyuk and V. V. Kabanov, *Stability of Shells* [in Russian], Nauka, Moscow (1978).
8. B. S. Garbow et al., *Matrix Eigensystem Routines–EISPACK Guide Extension*, Springer-Verlag, Berlin (1977). (Lect. Notes Comput. Sci., Vol. 51.)
9. Yu. V. Nemirovskii, "Theory of thermoelastic bending of reinforced shells and plates," *Mekh. Polimerov*, No. 5, 861–873 (1972).
10. A. N. Andreyev, "Stability of a layered cylindrical shell under external pressure," *Prikl. Mekh.*, **20**, No. 10, 59–64 (1984).
11. G. A. Vanin, N. P. Semenyuk, and R. F. Emel'yanov, *Stability of Shells from Reinforced Materials* [in Russian], Naukova Dumka, Kiev (1978).